Memory Aid Suggestions

Optimization :-

Graphing a line (functional) $y = -\frac{2}{3}x + 4$.

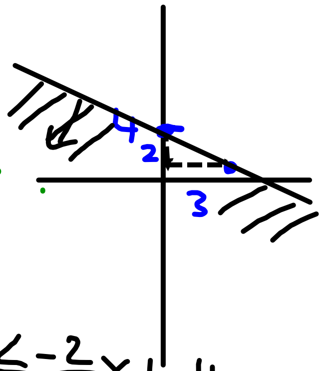
$y = ax + b$

slope. $\left(\frac{\text{Rise}}{\text{Run}}\right)$.

b → init value.

$<, \leq$ = shade Down

$>, \geq$ = " up



$y \leq -\frac{2}{3}x + 4$

Comparison Method.

$y = 2x + 4$

$y = -3x - 6$

$2x + 4 = -3x - 6$

$5x = -10$

$x = -2$

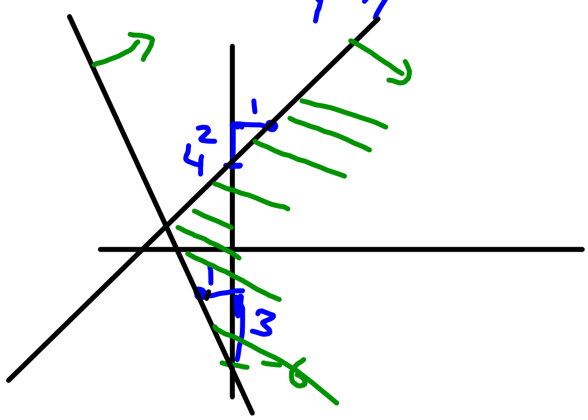
$(-2, 0)$.

$y = 2(-2) + 4$

$y = 0$.

$y \leq 2x + 4$

$y \geq -3x - 6$



Inequality Symbol	Meaning
<	"is less than" / "fewer than"
>	"is greater than" / "more than" / exceeds
≤	"is less than or equal to" / "at most" / "no more than" / " up to" / "maximum"
≥	"is greater than or equal to" / "at least" / "no less than" / "minimum"

Solving an OPTIMIZATION Problem

- Variables
- Constraints (Inequalities)
- Draw the Polygon of Constraints
- Vertices (x,y) of the Polygon of Constraints
- Optimization Rule
- Optimization Table
- Vertex (x,y) maximizing (minimizing) the function

Example:

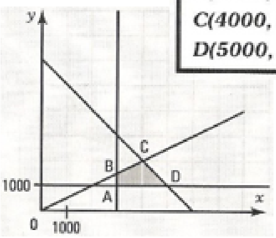
During the hockey playoffs, a shop owner decides to sell flags & caps with the Montreal Canadians logo. He orders 6000 items & expects to sell at least 3000 flags & at least 1000 caps. He also expects to sell at least twice as many flags as caps.

If the net profit on a flag is $15 and that on a cap is $12, how many flags & caps must he sell in order to maximize his profit?

Vertices	$R = 15x + 12y$
A(3000, 1000)	57 000
B(3000, 1500)	63 000
C(4000, 2000)	84 000
D(5000, 1000)	87 000

x: number of flags sold
y: number of caps sold

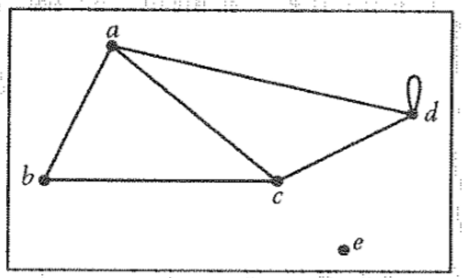
$x \geq 0$
$y \geq 0$
$x + y \leq 6000$
$x \geq 3000$
$y \geq 1000$
$x \geq 2y$



The shop owner must sell 5000 flags & 1000 caps.

- A graph allows us to illustrate the elements of a set using points called vertices and relationships between these elements using lines called edges.
- An edge between a vertex a and a vertex b is written $\{a,b\}$ or $\{b,a\}$ or simply ab or ba since the order in which the vertices are written is not important.
  An edge connecting a vertex to itself is called a loop of the graph.
- Two edges are called adjacent when they are connected by an edge.
- The number of vertices in a graph is called the order of the graph.
- The degree of a vertex is equal to the number of times it is reached by an edge.
  The degree of vertex A is written: $d(A)$.
- The degree of a vertex with a loop is at least equal to 2.

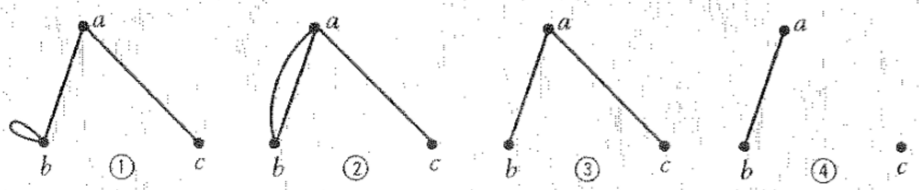
Ex.: Consider the set of vertices $S = \{a, b, c, d, e\}$
and the graph $G = \{ab, bc, ca, ad, cd, dd\}$.



- Vertices a and b are adjacent while vertices b and d are not adjacent.
- The order of the graph is equal to 5.
- The degree of vertex a is equal to 3, the degree of vertex d is equal to 4 (the loop at d counts as 2 edges), the degree of vertex e is equal to 0 because vertex e is isolated since there are no vertices in the graph adjacent to it.

SIMPLE GRAPH

A graph is called simple when it does not have loops and when each pair of vertices is connected by at most one edge.



– Graph ① is not simple because it has a loop at vertex *b*.

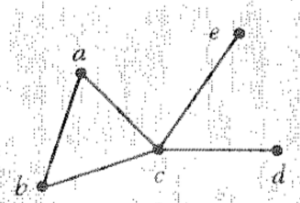
– Graph ② is not simple because vertices *a* and *b* are connected by 2 edges.

– Graph ③ and ④ are simple.

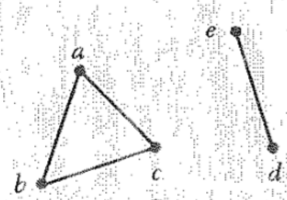
CONNECTED GRAPH

A graph is called connected if, between any two vertices of the graph, there exists an edge or a sequence of edges connecting these two vertices.

Ex.:



①



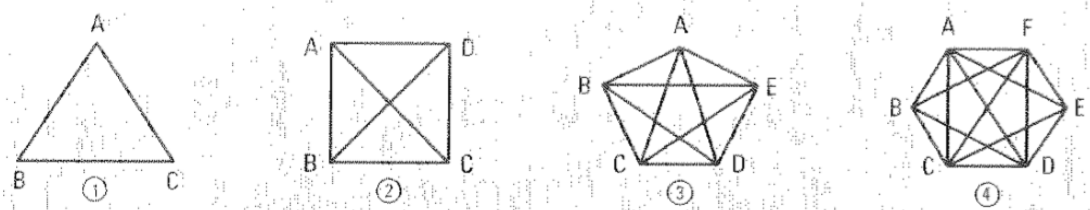
②

- Graph ① is connected.
- Graph ② is not connected because, for example, there is no edge or sequence of edges between vertices a and d.

COMPLETE GRAPH

A graph is called complete if, between any two vertices, there exists an edge connecting these two vertices.

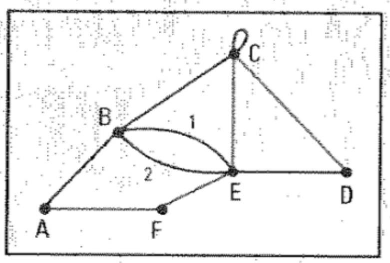
Ex.:



– Graphs ①, ② and ③ are complete.

– Graph ④ is not complete because edge BE does not exist.

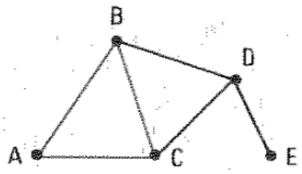
CHAIN – SIMPLE CHAIN



- A chain is a sequence of edges. The endpoints of the chain drawn in red on the right are A and D. It is written AFECD or DCEFA.
- The length of a chain is equal to the number of edges defining the chain. Chain AFECD has length 4.
- A chain can contain repeated vertices or edges. A chain is called simple if it does not contain repeated edges.
  - Chain AFECD is simple.
  - Chain AFECED is not simple because edge EC is repeated twice.
- The distance between two vertices A and B, written $d(A,B)$, is equal to the length of the shortest chain connecting these two vertices.
  Thus, we have: $d(A, B) = 1$, $d(A, C) = 2$, $d(A, D) = 3$.

CYCLE – SIMPLE CYCLE

- A chain beginning and ending at the same vertex is called a cycle.
  The chain represented in red is a cycle that can be named in 6 different ways: ABCA, BCAB, CABC, ACBA, CBAC and BACB. These 6 different ways of naming the chain define one and only one cycle, drawn in red.

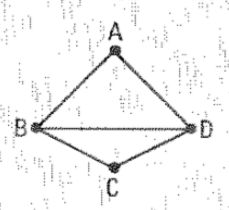


- A cycle is simple if it does not pass many times through the same edge.
  – ABCA is a simple cycle.
  – ABDCBA is not a simple cycle since it passes twice through edge AB.

EULERIAN CHAIN – EULERIAN CYCLE

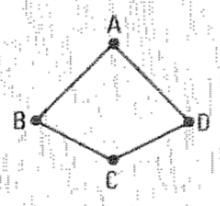
- In a connected graph, an Eulerian chain is a chain that passes through all edges of the graph once and only once. When an Eulerian chain begins and ends at the same vertex, it is called Eulerian cycle.
- There exists an Eulerian chain when the graph contains exactly two odd degree vertices. The odd degree vertices are then the beginning vertex and the ending vertex of the Eulerian chain.
- There exists an Eulerian cycle when all the vertices of the graph have even degree. The cycle can then begin at any vertex and end at this vertex.

Ex.:



BADCBD is an Eulerian chain.
deg(B) = 3 ; deg(D) = 3

Ex.:



ABCDA is an Eulerian cycle.
All vertices have even degree.

HAMILTONIAN CHAIN – HAMILTONIAN CYCLE

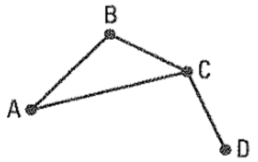
- In a connected graph, a Hamiltonian chain is a chain passing through all the vertices in the graph once and only once.

  When a Hamiltonian chain begins and ends at the same vertex, it is called Hamiltonian cycle.

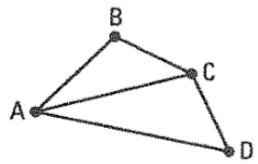
- A sufficient but not necessary condition for a connected graph having n vertices ($n \geq 2$) to contain a Hamiltonian cycle is that each vertex of the graph is connected to at least half the other vertices, in other words, each vertex has degree greater than or equal to $\frac{n}{2}$.

- Note that a graph with a vertex of degree 1 does not contain a Hamiltonian cycle.

  Ex.: The following graphs have 4 vertices. It suffices that all vertices have degree greater than or equal to 2 to have a Hamiltonian cycle.



- ABCD is a Hamiltonian chain.
- Since deg(D) = 1, there is no Hamiltonian cycle in this graph.

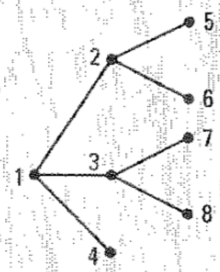


- ABCD is a Hamiltonian chain.
- All vertices have degree at least 2, so there exists a Hamiltonian cycle, namely ABCDA.

TREE

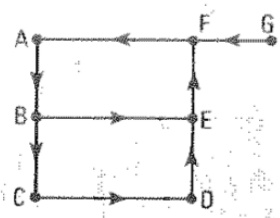
- A tree is a connected graph that does not contain a simple cycle.
- Any tree with n vertices has $(n - 1)$ edges.

Ex.: – The graph on the right is a tree.
– It has 8 vertices and 7 edges.



DIRECTED GRAPH

- A directed graph is a graph in which each edge is oriented. On each edge connecting two vertices, an arrow indicates the direction of motion between the two vertices.



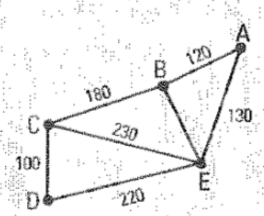
- In a directed graph, it is usual to call a directed edge an arrow, to call a directed chain a path and to call a directed cycle a circuit. We will not use these terms in order to simplify the language.

Ex.: In the directed graph on the right,
- edge AB exists and edge BA does not exist.
- the distance between vertices F and E is equal to 3 because the shortest chain FABE connecting vertices F and E contains 3 edges.
- the distance between vertices F and G does not exist because there is no chain connecting vertices F and G.
- there exist only two directed simple cycles: ABEFA and ABCDEFA.
- there exists only one directed Hamiltonian chain: GFABCDE.
- there does not exist any directed Eulerian cycle nor directed Hamiltonian cycle.

WEIGTHED GRAPH

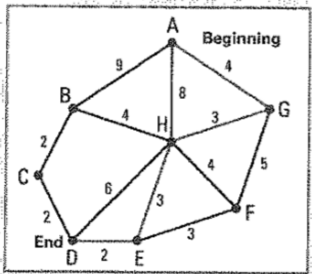
- A weighted graph is a graph where there are numerical values on the edges. These values can represent a distance, a duration, a cost, etc.
- The value of a chain is equal to the sum of the values of the edges that constitute the chain.
- A weighted graph can be directed or non directed.

Ex.: The graph on the right indicates the distance (in km) between different towns connected by a railroad.
The value of the chain ABCD is equal to 400 km and corresponds to the total distance traveled when we start from town A and travel to town D passing successively through towns B and C.



OPTIMAL CHAIN

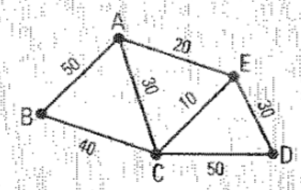
In the graph on the left, the chain with minimal value connecting beginning vertex A and ending vertex D is the chain AGHED with value 12.



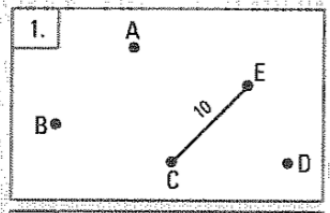
OPTIMAL TREE

Given a directed weighted graph, it is possible to find in this graph a weighted tree with minimal or maximal value.

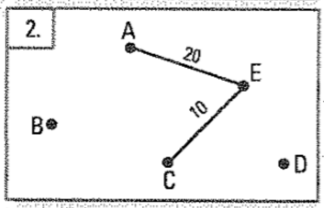
We use the following procedure to determine the minimal value tree.



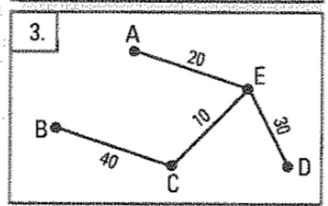
1. We draw the graph with only the vertices then we draw the edge with the lowest value.



2. We draw, as we go, the edge with the smallest value among the remaining edges while always avoiding adding an edge that forms a simple cycle.



3. We continue this procedure until we obtain a tree.



CRITICAL PATH

- A task requiring the completion of several steps can be represented by a directed weighted graph taking into account
  - the prerequisite steps,
  - the steps that can be carried out simultaneously.
- The critical path is the directed chain in the graph that has maximal value.

  The value of the critical path represents the minimum time for the completion of all the steps that constitute the task.

  Ex.: See activity 1.

  The critical path Beginning ABEFGHIJM End has value 73.

  73 days are necessary for the production and distribution of the book.

GRAPH COLORING

- When solving a problem, it can prove useful to color the vertices of a graph. The coloring rules are as follows:
  1. Two adjacent vertices always have different colors.
  2. A minimal number of different colors must be used to color the vertices of the graph.

In practice, after ordering the vertices in decreasing order of degree, we color the vertex with highest degree then we color, in order, the other vertices while respecting the two coloring rules. We therefore reuse, as soon as possible, the colors that have already been used.

Ex.: Some students in class are talkative. The teacher has made the graph below. Anita, Beatrix, Celia, Daniel, Evelyn, Frank and Georges are represented by the graph's vertices. He connects two vertices (students) in the graph with an edge to indicate that the two students must not sit next to each other.

- The chromatic number of the graph is 3.
  The teacher can seat Georges, Celia, Beatrix and Evelyn together, Anita and Frank together while Daniel will be isolated from the other students mentioned.